Assignment III MTH 512 , Fall 2018

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QUESTION 1. Let V be a a vector space over the field R. Assume $IN(V)$ is and odd number ≥ 3 (i.e., $dim(V)$ is an odd integer ≥ 3). Assume $T : V \to V$ is a linear transformation. Convince me that there is a nonzero element in V, say v, and a real number $c \in R$ such that $T(v) = cv$. (short proof)

$IN(V) = 2n+1 \quad n \in \mathbb{N}^* = \{1,2,3,--\}$

Let M be the standard matrix rep. of T , M is $(2n+1) X (2n+1)$ $n \in \mathbb{N}$

ok $C_M(\alpha) = |\alpha I_{2n+1} - M|$ is a polynomial of degree $2n+1$

$=$ product of polynomials of degree 1 or degree 2

$\Rightarrow C_M(\alpha)$ has at least one real root $(C \in \mathbb{R})$ (since degree is odd ≥ 3.

$\Rightarrow \exists C \in \mathbb{R}$ s.t. $|C I_{2n+1} - M| = 0 \Rightarrow C$ is an eigenvalue of V under M

(eigenvalue) $\Rightarrow \exists v \in V \; (v \neq 0_v)$ s.t $cv = Mv \overset{\text{Translate}}{\Longrightarrow} T(v) = cv$

QUESTION 2. Let $T : R^3 \to R^3$ such that $T(a_1, a_2, a_3) = (0, a_1+2a_3, a_2-a_3)$. Then clearly T is a linear transformation (do not show that).

- Convince me that R^3 has exactly 3 eigenspaces under T and construct such subspaces.

Let M be the standard matrix representation of T , M is 3x3

ok

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \qquad |\alpha I_3 - M| = \begin{vmatrix} \alpha & 0 & 0 \\ -1 & \alpha & -2 \\ 0 & -1 & \alpha+1 \end{vmatrix}$$

$$= (-1)^2 \alpha \begin{vmatrix} \alpha & -2 \\ -1 & \alpha+1 \end{vmatrix} = \alpha[\alpha^2 + \alpha - 2] = \alpha(\alpha-1)(\alpha+2)$$

set $|\alpha I_3 - M| = 0 \Rightarrow$ Eigen values are $\boxed{\alpha_1 = 0 , \alpha_2 = 1 , \alpha_3 = -2 \in \mathbb{R}}$

$\Rightarrow \mathbb{R}^3$ has exactly 3 eigen spaces under T (we have 3 eigenvalues in $\mathbb{R}$)

$\underline{\alpha_1 = 0} \Rightarrow$ solve homogeneous $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} -a_2 + a_3 = 0 \\ -a_1 - 2a_3 = 0 \end{array} \quad \begin{array}{l} a_2 = a \\ a_1 = -2a \end{array}$

✓ $\Rightarrow E_0 = \{(-2a_3, a_3, a_3)\} = \{a_3(-2,1,1)\} = \text{Span}\{(-2,1,1)\}$

$\underline{\alpha_2 = 1} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \underset{\sim}{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \underset{\sim}{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow a_1 = 0 \quad , \quad a_2 - 2a_3 = 0 \quad a_2 = 2a_3$

✓ $E_1 = \{(0, 2a_3, a_3)\} = \{a_3(0,2,1)\} = \text{Span}\{(0,2,1)\}$

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Question 2 : (Continued)

$$\alpha = -2 \quad \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \underset{\substack{-R_2 \to R_2 \\ -R_3 \to R_3}}{\overset{-\frac{1}{2}R_1 \to R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{-R_1 + R_2 \to R_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{\frac{1}{2}R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{-R_2 + R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow a_1 = 0 \qquad a_2 = -a_3$

$$E_{-2} = \{(0, -a_3, a_3)\} = \{a_3(0, -1, 1)\} = \text{span}\{(0, -1, 1)\}$$

✓

- Convince me that T is singular (i.e., T is not invertible) (Short sentence)

OK $|M| = 0 \Rightarrow M$ is not invertible $\Rightarrow T$ is not invertible

- Construct a diagonal linear transformation, say F (from R^3 into R^3), and construct a nonsingular (invertible) linear transformation L (from R^3 into R^3) such that $L \circ F \circ L^{-1} = T$. (Do not construct L^{-1})

OK $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $Q = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $M = QDQ^{-1}$

$T = L \circ F \circ L^{-1}$ ✓

$L : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $T(a_1, a_2, a_3) = Q\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ ✓

$F(a_1, a_2, a_3) = a_1(0,0,0) + a_2(0,1,0) + a_3(0,0,-2) = (0, a_2, -2a_3)$ ✓

$F : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $F(a_1, a_2, a_3) = D\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ ✓

$L(a_1, a_2, a_3) = (-2a_1, a_1 + 2a_2 - a_3, a_1 + a_2 + a_3)$

QUESTION 3 (Hint: Short answers! Not much work, use class notes and stare at Question 2). . Let $L : P_3 \to P_3$ such that $L(ax^2 + bx + c) = (a + 2c)x + b - c$. Clearly that L is a linear transformation

- Convince me that P_3 has exactly 3 eigenspaces under L and construct such subspaces.

$P_3 \approx \mathbb{R}^3$ $L'(a,b,c) = (0, a+2c, b-c)$ $L' : \mathbb{R}^3 \to \mathbb{R}^3$

S.M.R of $L' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = M$ (in question 2)

$\Rightarrow P_3$ has 3 eigenvalues under L, $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = -3 \in \mathbb{R}$

OK $\Rightarrow P_3$ has 3 eigenspaces under L.

$E_0 = \text{span}\{-2x^2 + x + 1\}, E_1 = \text{span}\{2x+1\}, E_{-2} = \text{span}\{-x+1\}$

- Convince me that L is singular (i.e., L is not invertible) (Short sentence)

$|M| = 0 \Rightarrow M$ is not invertible $\Rightarrow T$ is not invertible

- Construct a diagonal linear transformation, say F (from P_3 into P_3), and construct a nonsingular (invertible) linear transformation $\underset{K}{L}$ (from P^3 into P_3) such that $\underset{K}{L} \circ F \circ \underset{K^{-1}}{L^{-1}} = \underset{L}{T}$. (Do not construct $\underset{K}{L^{-1}}$)

using Q2 $\Rightarrow F'(a,b,c) = (0, b, -2c) \Rightarrow F(ax^2+bx+c) = bx - 2c$ ✓

$K(a,b,c) = (-2a, a+2b-c, a+b+c)$

OK $\Rightarrow L(ax^2+bx+c) = -2ax^2 + (a+2b-c)x + (a+b+c)$ ✓

- Let $D = \{f(x) \in P_3 \mid L(f(x)) = 3x + 1\}$. Describe the elements in D

$P_3 \approx \mathbb{R}^3$, $L' : \mathbb{R}^3 \to \mathbb{R}^3$

$L'(a,b,c) = (0, a+2c, b-c)$

$D' = \{(a,b,c) \mid L'(a,b,c) = (0,3,1)\}$

OK $= \{(a,b,c) \mid (0, a+2c, b-c) = (0,3,1)\}$

Note: I used the original L of the question
$L(ax^2+bx+c) = (a+2c)x + b - c$

$= \{(a,b,c) \mid a + 2c = 3 \text{ and } b - c = 1\}$

$= \{(3-2c, c+1, c) \mid c \in \mathbb{R}\} = \{(3,1,0) + c(-2,1,1) \mid c \in \mathbb{R}\}$

$\Rightarrow D = \{3x^2 + x + c(-2x^2 + x + 1) \mid c \in \mathbb{R}\}$

QUESTION 4. (i) Let A, $n \times n$, be a nonsingular matrix over a field F. Suppose that c is an eigenvalue of A. Convince me that c^{-1} is an eigenvalue of A^{-1} and $E_c = E_{c^{-1}}$ (note that c^{-1} is the inverse of c under multiplication) (short ANSWER)

c is an eigenvalue of A ⇒ A has an eigen space E_c

$Av = cv \quad \forall v \in E_c$

Since A is invertible ⇒ A^{-1} exists

⇒ $A^{-1}Av = A^{-1}cv = cA^{-1}v \quad \forall v \in E_c$

$I_n v = cA^{-1}v$

$v = cA^{-1}v$

$c^{-1}v = c^{-1}cA^{-1}v = 1A^{-1}v = A^{-1}v$

OK

⇒ $c^{-1}v = A^{-1}v$ $\forall v \in E_c$ ⇒ c^{-1} is an eigenvalue of A^{-1} and $v \in E_c$

⇒ $E_c = E_{c^{-1}}$

(ii) Assume that A is a 3×3 matrix with eigenvalues 2, 5, 3. Find $C_{A^{-1}}(\alpha)$ (Very short answer!)

By (i) $\alpha_1 = 1/2$, $\alpha_2 = 1/5$, $\alpha_3 = 1/3$ are eigenvalues of A^{-1} (3x3)

⇒ $C_{A^{-1}}(\alpha)$ is a polynomial of degree 3 with roots $\alpha_1, \alpha_2, \alpha_3$

$C_{A^{-1}}(\alpha) = (x - 1/2)(x - 1/5)(x - 1/3)$

OK

(iii) Let A, $n \times n$, be matrix over a field F. Suppose that c is an eigenvalue of A. Convince me that c^n is an eigenvalue of A^n for every positive integer $n \geq 2$.

A nxn, c is an eigenvalue of A ⇒ $Av = cv$ $v \in E_c$ (eigen space of

required to prove that $A^n v = c^n v$

we proceed by induction

n=1 $Av = cv$ (given, since c is an eigenvalue of A)

Assume $A^{n-1}v = c^{n-1}v$

Proof for n: $A^n v = AA^{n-1}v = Ac^{n-1}v = c^{n-1}Av = c^{n-1}cv = c^n v$

$n \geq 2$

⇒ $A^n v = c^n v$

OK

⇒ c^n is an eigenvalue of A^n ($n \geq 2$)

Brief Lecture: From this question we learn the following. Let V be a finite dimensional vector space over a field F. Assume that $T : V \to V$ is a linear transformation. If T is 1-1 and onto and c is an eigenvalue of T, then c^{-1} is an eigenvalue of T^{-1} and $E_c = E_{c^{-1}}$. If c is an eigenvalue of T, then c^n is an eigenvalue of T^n for every positive integer $n \geq 2$. (note: $T^n = T \circ T \circ \cdots \circ T$)

QUESTION 5. (Short answer, but think) Let A be a 4×4 matrix over R. Given $|A| = 30$, $3, 5$ are eigenvalues of A, and Trace(A) = 10. Convince me that A is not diagnolizable over $\underline{R}$. Find $C_A(\alpha)$. Is A diagnolizable over C? explain BRIEFLY.

$C_A(\alpha)$ is a polynomial of degree 4 $\Rightarrow$ there are 4 eigenvalues of A in $\mathbb{C}$ $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$\alpha_1 = 3$, $\alpha_2 = 5$

$|A| = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 = 30 \Rightarrow \alpha_3 \cdot \alpha_4 = 2 \Rightarrow \alpha_4 = 2/\alpha_3$

$\text{Trace}(A) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 10 \Rightarrow \alpha_3 + \alpha_4 = 2 \Rightarrow \alpha_3 + 2/\alpha^3 = 2 \Rightarrow \alpha_3^2 - 2\alpha_3 + 2 = 0$

$\longrightarrow$ see back of the page

QUESTION 6. Let $T : R^4 \to R^3$ such that $T(a_1, a_2, a_3, a_4) = (a_1 + a_2 + a_3, a_3 + 2a_4, -a_1 - a_2 - a_3)$

(i) Find the standard matrix representation of T.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

(ii) Let $B = \{\overset{v_1}{(1,0,0,0)}, \overset{v_2}{(0,1,-1,-1)}, \overset{v_3}{(0,0,1,1)}, \overset{v_4}{(0,1,-1,0)}\}$ be a basis for R^4, and $B' = \{\overset{w_1}{(1,-1,0)}, \overset{w_2}{(0,1,-1)}, \overset{w_3}{(1,-1,1)}\}$ be a basis for R^3. Find the matrix representation of T with respect to B and B'.

$$Q = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad W = \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \qquad \underset{\llcorner \text{ matrix rep. of T w.r.t B\&B'}}{M_{BB'}} = W^{-1} M Q$$

$$\left[\underbrace{\begin{matrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{matrix}}_{W} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right] \underset{\sim}{R_1 + R_2 \rightarrow R_2} \left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{matrix}\right] \underset{\sim}{R_2 + R_3 \rightarrow R_3}$$

$$\left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}\right] \underset{\sim}{-R_3 + R_1 \rightarrow R_1} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}}_{W^{-1}}\right]$$

$$M_{BB'} = \overset{W^{-1}}{\begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \overset{M}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{bmatrix}} \overset{Q}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} = \underset{\substack{M_{BB'} \\ \text{matrix rep of T w.r.t B\&B'}}}{\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & -3 & 4 & -1 \\ 0 & -3 & 3 & -1 \end{bmatrix}} \checkmark$$

Q 5) continued

Polynomial of degree 2!

$\alpha_3^2 - 2\alpha_3 + 2 = 0$ has no real roots (can't be reduced in $\mathbb{R}$)

$\Rightarrow \alpha_3 \in \mathbb{C} \Rightarrow \alpha_4 = \overline{\alpha_3}$ (complex conjugate)

* $\Rightarrow$ A is not diagonalizable over $\mathbb{R}$ $\rightarrow$ because $C_A(\alpha)$ is not product of linear factors

** $C_A(\alpha) = (x-3)(x-5)(x-\alpha_3)(x-\overline{\alpha_3})$

where α_3 is the root of $\alpha_3^2 - 2\alpha_3 + 2 = 0$, $\alpha_3 \in \mathbb{C}$

*** Yes, A is diagonalizable over $\mathbb{C}$

Because the power of all Linear factors of $C_A(\alpha)$ is 1

* as per class notes, there's no need to check that $IN(E_{\alpha_i}) = n_i$ where n_i is the power of the linear factor $(x-\alpha_i)$ since all powers are 1

~~Outside the~~ In the scope of the course

$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 = 30$

$\alpha_3 \cdot \alpha_4 = 2 \qquad \alpha_4 = \overline{\alpha_3}$

$\Rightarrow Re^2(\alpha_3) + Im^2(\alpha_3) = 2$

OK

$\alpha_3 + \alpha_4 = 2$

$2Re(\alpha_3) = 2$

$Re(\alpha_3) = 1 \Rightarrow 1 + Im^2(\alpha_3) = 2$

$Im(\alpha_3) = 1$

$\Rightarrow \alpha_3 = 1 + i \qquad \alpha_4 = 1 - i$

satisfies $\alpha_3^2 - 2\alpha_3 + 2 = 0$

$\alpha_3 = 1 + i$

$\alpha_4 = \overline{\alpha_3} = 1 - i$

(iii) Find a general formula for $[(a_1, a_2, a_3, a_4)]_B$

$$Q^{-1}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \qquad \left[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{matrix}}_{Q} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I_4} \right] \begin{matrix} R_2+R_3 \Rightarrow R_3 \\ \sim \\ R_2+R_4 \Rightarrow R_4 \end{matrix}$$

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$$[(a_1, a_2, a_3, a_4)]_B = (a_1, a_2+a_3-a_4, a_2+a_3, -a_3+a_4)$$ OK

(iv) Find $[(2, 4, 1, 1)]_B$

$= (2, 4, 5, 0)$ ✓ OK

(v) Use the matrix in (i) and find $T(2, 4, 1, 1)$

$$= 2(1,0,-1) + 4(1,0,-1) + 1(1,1,-1) + 1(0,2,0)$$

$$= (2+4+1, 1+2, -2-4-1) = (7, 3, -7)$$ ✓ OK

(vi) Use the matrix in (*ii*) and find $T(2, 4, 1, 1)$ (show the steps) using $M_{BB'}$

$M_{BB'}\begin{pmatrix} 2 \\ 4 \\ 5 \\ 0 \end{pmatrix}$

$$[(2,4,1,1)]_B = (2,4,5,0)$$

$$[T(2,4,1,1)]_{B'} = 2(1,1,0) + 4(3,-3,-3) + 5(-2,4,3)$$

$$= (2+12-10, 2-12+20, -12+15) = (4, 10, 3)$$ ✓

$$T(2,4,1,1) = 4\underset{w_1}{(1,-1,0)} + 10\underset{w_2}{(0,1,-1)} + 3\underset{w_3}{(1,-1,1)} = (4+3, -4+10-3, -10+$$

$$= (7, 3, -7)$$

(vii) Find $Z(T)$ (i.e., Ker(T)). [It is much easier to use the matrix in (i)]

$$M = \left[\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \underset{\sim}{R_1+R_3 \Rightarrow R_3} \left[\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \underset{\sim}{-R_2+R_1 \Rightarrow R_1}$$

$$\left[\begin{matrix} \textcircled{1} & 1 & 0 & -2 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$a_1 + a_2 - 2a_4 = 0$

$\Rightarrow a_1 = 2a_4 - a_2$

$a_3 + 2a_4 = 0$

$a_3 = -2a_4$

$$Z(T) = \{(2a_4 - a_2, a_2, -2a_4, a_4)\}$$

$$= \{(-a_2, a_2, 0, 0) + (2a_4, 0, -2a_4, a_4)\}$$

$$= \{a_2(-1,1,0,0) + a_4(2,0,-2,1)\}$$

$$= \text{span}\{(-1,1,0,0), (2,0,-2,1)\}$$

iii) continued

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_4 \to R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_4+R_2 \to R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{Q^{-1}}$

$$[(a_1, a_2, a_3, a_4)]_B = a_1(1,0,0,0) + a_2(0,1,1,0) + a_3(0,1,1,-1)$$

$$+ a_4(0,-1,0,1) = (a_1, a_2+a_3-a_4, a_2+a_3, -a_3+a_4)$$

OK

(viii) Find a general formula for $[(a_1, a_2, a_3)]_{B'}$

$$W^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$[(a_1, a_2, a_3)]_{B'}$

$= a_1(0,1,1) + a_2(-1,1,1) + a_3(-1,0,1)$

$= (-a_2 - a_3, a_1 + a_2, a_1 + a_2 + a_3)$ ✓

(ix) Find a basis for the Range of T, Say, $D = \{W_1, W_2, \ldots, W_k\}$ is such basis. Then find $[W_i]_{B'}$ for each $W_i \in D$.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{bmatrix} \text{ S.M.R of } T$$

$\text{Range}(T) = \text{span}\{(1,0,-1), (1,1,-1)\}$ ✓

$D = \{\underbrace{(1,0,-1)}_{W_1}, \underbrace{(1,1,-1)}_{W_2}\}$ ✓

using (viii)

$[W_1]_{B'} = [(1,0,-1)]_{B'} = (1,1,0)$

$[W_2]_{B'} = [(1,1,-1)]_{B'} = (0,2,1)$ ✓

QUESTION 7. (Not much work!, stare at question 6!, almost done!) Let $L : R^{2\times 2} \to P^3$ such that $L(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}) = (a_1 + a_2 + a_3)x^2 + (a_3 + 2a_4)x + -a_1 - a_2 - a_3$

(i) Find the fake standard matrix representation of L. $\mathbb{R}^{2\times 2} \approx \mathbb{R}^4$ $P^3 \approx \mathbb{R}^3$

$L'(a_1, a_2, a_3, a_4) = (a_1 + a_2 + a_3, a_3 + 2a_4, -a_1 - a_2 - a_3)$

(same as q6.) ⟶ $M' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{bmatrix}$ (3x4) ok

(ii) Let $B = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$ be a basis for $R^{2\times 2}$, and $B' = \{x^2 - x, x - 1, x^2 - x + 1\}$ be a basis for P_3. Find the fake matrix representation of L with respect to B and B'.

From question 6: $\mathbb{R}^{2\times 2} \approx \mathbb{R}^4$ $P^3 \approx \mathbb{R}^3$

$M'_{BB'} = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & -3 & 4 & -1 \\ 0 & -3 & 3 & -1 \end{bmatrix}$ ✓

(iii) Find a general formula for $[\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}]_B$ =

Exact ANSWER AS in 6(iii)

using question 6 (iii)

ok

(iv) Find $[\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}]_B$ =

(2, 4, 5, 0)
EXACT
ANSWER as in
6(iii)

using (iii)

ok

(v) Use the matrix in (i) and find $L(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}) = 7x^2 + 3x - 7$

$L'(2, 4, 1, 1) = (7, 3, -7)$

ok

(vi) Use the matrix in (ii) and find $L(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix})$ (show the steps)

$L'(2,4,1,1)$ using $M'_{BB'}$

$[(2,4,1,1)]_B = (2,4,5,0)$

$[L'(2,4,1,1)]_{B'} = (4,10,3) \Rightarrow L'(2,4,1,1) = (7,3,-7)$

$\longrightarrow$ using $M'_{BB'}$ (question 6)

$\Rightarrow L(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}) = 7x^2 + 3x - 7$ OK

(vii) Find $Z(L)$ (i.e., Ker(L)). [It is much easier to use the matrix in (i)]

$Z(L') = \text{span}\{(-1,1,0,0), (2,0,-2,1)\}$

$Z(L) = \text{span}\{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\}$ (From Q6) OK

(viii) Find a general formula for $[\underset{a_1}{a_2}x^2 + \underset{a_2}{a_1}x + \underset{a_3}{a_0}]_{B'}$

$= \Big((-a_1 - a_0), \cancel{x^2} \cancel{+} (a_2 + a_1) \cancel{x}, \cancel{+} (a_2 + a_1 + a_0) \Big)$ (From Q6)

OK

(ix) Find a basis for the Range of L, Say, $D = \{W_1, W_2, ..., W_k\}$ is such basis. Then find $[W_i]_{B'}$ for each $W_i \in D$.

$\text{Range}(L) = \text{span}\{x^2 - 1, x^2 + x - 1\}$

$[W_1]_{B'} = [x^2 - 1]_{B'} =$

Same answer as Question 6 Viii

(using viii) & Q6

$[W_2]_{B'} = [x^2 + x - 1]_{B'}$

OK

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